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LETTER TO THE EDITOR

Segment distribution in self-similar patterns and backbone structure of percolation clusters

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Abstract. A distribution function of segments in self-similar patterns is investigated and determined from the fractal dimensionality D of the pattern and that D' of the segments. On the basis of scaling assumptions, the critical behaviour of the segment distribution is studied. Critical exponents are introduced and expressed in terms of D and D'. The general theory is applied to link distribution in the backbone of an infinite percolation cluster. Critical exponents for link distribution are also introduced and evaluated explicitly.

The concept of self-similarity is widely used in the physical sciences. Various types of random structures in nature are self-similar and described as fractals (Mandelbrot 1977, 1982). There frequently exist typical segments dominating the physical properties of self-similar patterns. Therefore, the understanding of segment distribution is significant in the physics of self-similar systems. Recently, Sawada *et al* (1982) discussed the distribution of one-dimensional branches in a random pattern. In this letter, we extend their treatment to general segment distribution in self-similar patterns. The critical behaviour associated with segment distribution is investigated on the basis of scaling assumptions, which is the first purpose of this letter. In percolation theory, study of the backbone structure of an infinite cluster has received considerable attention in connection with many physical properties of random systems such as conductivity of random resistor networks and magnetic order in dilute ferromagnets (Skal and Shklovskii 1974, de Gennes 1976, Stanley 1977, Kirkpatrick 1978, Coniglio 1982). Thus, a second purpose of this letter is to apply the general theory to link distribution in the cluster backbone and to clarify the geometrical picture of the backbone.

We consider a self-similar pattern with a fractal (Hausdorff) dimensionality D which consists of segments with a fractal dimensionality D' less than D. By definition of the fractal (Hausdorff) dimension (Mandelbrot 1977), the minimum number $N(\eta)$ of balls with a radius η which are necessary to cover the whole pattern is proportional to η^{-D}:

$$N(\eta) \propto \eta^{-D}. \tag{1}$$

Similarly, the number $n_l(\eta)$ of balls necessary to cover a segment of length l is given by

$$n_l(\eta) \propto (l/\eta)^{D'}. \tag{2}$$

On the other hand, $N(\eta)$ and $n_l(\eta)$ satisfy the relation (Sawada *et al* 1982)

$$N(\eta) \simeq \int_\eta^\infty n_l(\eta) f(l)\, \mathrm{d}l, \tag{3}$$

where $f(l)$ is a distribution function of segments of length l. From (1)–(3), we find

$$f(l) \propto l^{-(1+D)}. \tag{4}$$

The discussion is extended to a quasi-self-similar pattern, i.e. a pattern which consists of unit components of length unity and is self-similar only in the length scale $1 \ll L \lesssim \xi$. We call a segment composed of s unit components an s-unit segment and define a distribution function $\rho(s, \xi)$ of s-unit segments by

$$\rho(s, \xi) \equiv \frac{\text{number of } s\text{-unit segments}}{\text{number of units belonging to the whole pattern}}. \tag{5}$$

In the length scale $1 \ll L \lesssim \xi$, the length l_s of an s-unit segment is expressed as

$$l_s^{D'} \propto s \tag{6}$$

and $\rho(s, \xi)$ is related to $f(l_s)$ as

$$\rho(s, \xi)\, \mathrm{d}s = f(l_s)\, \mathrm{d}l_s. \tag{7}$$

Substitution of (4) and (6) into (7) leads to

$$\rho(s, \xi) \propto s^{-(1+D/D')} \qquad (1 \ll s \lesssim \xi^{D'}). \tag{8}$$

On the basis of scaling assumptions, we investigate critical behaviour about segment distribution. Here $\rho(s, \xi)$ is assumed to have the scaling form (Stauffer 1979)

$$\rho(s, \xi) = s^{-\tau} F(s/s_\xi), \tag{9}$$

where s_ξ is a typical segment size dominating all critical phenomena and defined by

$$s_\xi \propto \xi^{1/\hat{\sigma}}. \tag{10}$$

Comparing (9) and (10) with (6) and (8), we have

$$\hat{\sigma} = 1/D', \tag{11}$$

$$\tau = 1 + D/D'. \tag{12}$$

We introduce critical exponents describing the segment distribution defined by

$$\left[\sum_s \rho(s, \xi)\right]_{\text{sing}} \propto \xi^{-\hat{\alpha}}, \tag{13}$$

$$\left[\sum_s s\rho(s, \xi)\right]_{\text{sing}} \propto \xi^{-\hat{\beta}}, \tag{14}$$

$$\left[\sum_s s^2\rho(s, \xi)\right]_{\text{sing}} \propto \xi^{\hat{\gamma}}, \tag{15}$$

$$\left[\sum_s s\rho(s, \xi)\, \mathrm{e}^{-hs}\right]_{\text{sing}} \propto h^{1/\delta}, \tag{16}$$

where Σ_s denotes the sum over all segments and the subscript 'sing' means the singular part, namely, the leading non-analytic part of the subscripted quantity. Substitution of (9)–(12) into (13)–(16) gives (for details, see Stauffer 1979)

$$\left[\sum_s s^k \rho(s, \xi)\right]_{\text{sing}} \propto \xi^{(\tau-1-k)/\hat{\sigma}} = \xi^{D-kD'}, \tag{17}$$

$$\left[\sum_s s\rho(s, \xi)\, \mathrm{e}^{-hs}\right]_{\text{sing}} \propto h^{\tau-2} = h^{D/D'-1}. \tag{18}$$

Then we find

$$\hat{\alpha} = D, \qquad \hat{\beta} = D - D', \tag{19), (20}$$

$$\hat{\gamma} = 2D' - D, \qquad \delta = D'/(D - D'). \tag{21), (22}$$

We now specialise to the backbone of the infinite cluster in percolation lattices. Here the backbone is regarded as composed of one-dimensionally connected links defined as follows. In the length scale less than coherence length ξ, the backbone consists of singly connected channels and multiply connected blobs (Stanley 1977, Coniglio 1982). (Coniglio called this singly connected part 'links'. This definition of links is different from ours presented here.) If two properly chosen bonds in channels are cut, an isolated cluster is separated from the backbone. We call the component separable from the other part by cutting only two bonds as a biconnected cluster. Under the condition that each cluster is formed as large as possible, we can divide the backbone into biconnected clusters uniquely. In each biconnected cluster, an s-site (-bond) link is defined by the largest one-dimensionally connected chain containing s sites (bonds). In the remaining part of the cluster, an s'-site link is similarly defined by the largest chain. Iterating this procedure until all sites in the backbone are assigned to links, we can define link distribution uniquely.

In this case, a distribution function $\rho_{\mathrm{L}}(s, p)$ of s-site links is defined by

$$\rho_{\mathrm{L}}(s, p) \equiv \frac{\text{number of } s\text{-site links}}{\text{number of sites belonging to the backbone}}, \tag{23}$$

where p is a percolation probability. Associated critical exponents are given by

$$\rho_{\mathrm{L}}(s, p) = s^{-\tau_{\mathrm{L}}} F_{\mathrm{L}}(s/s_\xi), \tag{24}$$

$$s_\xi \propto (p - p_{\mathrm{c}})^{-1/\sigma_{\mathrm{L}}} \simeq \xi^{1/(\sigma_{\mathrm{L}}\nu)}, \tag{25}$$

$$\left[\sum_s \rho_{\mathrm{L}}(s, p)\right]_{\text{sing}} \propto (p - p_{\mathrm{c}})^{2-\alpha_{\mathrm{L}}} \simeq \xi^{-(2-\alpha_{\mathrm{L}})/\nu}, \tag{26}$$

$$\left[\sum_s s\rho_{\mathrm{L}}(s, p)\right]_{\text{sing}} \propto (p - p_{\mathrm{c}})^{\beta_{\mathrm{L}}} \simeq \xi^{-\beta_{\mathrm{L}}/\nu}, \tag{27}$$

$$\left[\sum_s s^2\rho_{\mathrm{L}}(s, p)\right]_{\text{sing}} \propto (p - p_{\mathrm{c}})^{-\gamma_{\mathrm{L}}} \simeq \xi^{\gamma_{\mathrm{L}}/\nu}, \tag{28}$$

$$\left[\sum_s s\rho_{\mathrm{L}}(s, p)\, \mathrm{e}^{-hs}\right]_{\text{sing}} \propto h^{1/\delta_{\mathrm{L}}}, \tag{29}$$

where p_{c} is the percolation threshold and ν is the critical exponent for the coherence length ξ defined by $\xi \propto (p - p_{\mathrm{c}})^{-\nu}$. In the length scale $1 \ll L \leqslant \xi$, the cluster backbone is self-similar and has a fractal dimensionality $d_{\mathrm{B}} = d - \beta_{\mathrm{B}}/\nu$ (Kirkpatrick 1978). On the other hand, one-dimensionally connected links are considered to make self-avoiding walks (Stanley 1977). Then the fractal dimensionality d_L of links is given by

$$d_L = \nu_{SAW}^{-1}, \tag{30}$$

where ν_{SAW} is the exponent for self-avoiding walks defined by the mean-square end-to-end distance $\langle R^2 \rangle$ and the number of steps N as $\langle R^2 \rangle \propto N^{2\nu_{SAW}}$. On the basis of the general theory, we can express all critical exponents in terms of d_B and d_L:

$$(2-\alpha_L)/\nu = d_B, \qquad \beta_L/\nu = d_B - d_L, \tag{31), (32}$$

$$\gamma_L/\nu = 2d_L - d_B, \qquad \delta_L = d_L/(d_B - d_L), \tag{33), (34}$$

$$\sigma_L \nu = 1/d_L, \qquad \tau_L = 1 + d_B/d_L. \tag{35), (36}$$

Explicit values of critical exponents evaluated from known estimates for ν and β_B are listed in table 1. For ν_{SAW}, we adopt the Flory formula (Flory 1953), $\nu_{SAW} = 3/(d+2)$ $(d \leq 4)$ and 0.5 $(d \geq 4)$.

Table 1. Critical exponents for link distribution.

d	ν[a]	β_B[b]	ν_{SAW}[d]	α_L	β_L	γ_L	δ_L	σ_L	τ_L
1	1	0	1	1	0	1	∞	1	2
2	1.35	0.5	0.75	−0.2	0.4	1.4	4.5	0.6	2.2
3	0.84	0.9	0.6	0.4	0.2	1.2	6.4	0.7	2.2
4	0.7	1.1	0.5	0.3	0.3	1.1	4.7	0.7	2.2
5	0.6		0.5						
6	0.5	2[c]	0.5	1	0	1	∞	1	2

[a] Stauffer (1979).
[b] Kirkpatrick (1978).
[c] Stanley (1977).
[d] Flory (1953).

Recently, Ohtsuki and Keyes (1984a) have introduced and calculated the fractal dimensionality d_L^* for the backbone length from a dynamical point of view. The fractal dimensionality d_L represents the length along the one-dimensionally connected links in the backbone, whereas d_L^* shows the effective length along the whole backbone. Hence the relation between d_L and d_L^* is not trivial. However, explicit estimates of d_L^* suggest that d_L^* is nearly equal to d_L (Ohtsuki and Keyes 1984a). If $d_L = d_L^*$, we can get the expression of the dynamical exponent μ for the conductivity in terms of the static exponents ν, β_B and ν_{SAW}:

$$\mu = 2\nu(\nu_{SAW}^{-1} - 1) + \beta_B. \tag{37}$$

Equations (11), (12) and (19)–(22) hold generally in self-similar systems. The random pattern discussed by Sawada *et al* (1982) is one example with $D' = 1$. In percolation lattices, critical exponents for finite cluster distribution in the whole lattices and dead end distribution in the infinite cluster are all described by them (Stauffer 1979, Ohtsuki and Keyes 1984b). The infinite cluster has a hierarchical structure, i.e. the cluster consists of dead ends and dead ends are composed of links. The present arguments are also applicable to such systems, e.g. link distribution in the whole infinite cluster including dead ends. Consider a hierarchical pattern where segments consist of subsegments with a fractal dimensionality D'' less than D'. Let $\rho'(t, \xi)$ be a distribution function of t-unit subsegments defined by

$$\rho'(t, \xi) \equiv \frac{\text{number of } t\text{-unit subsegments in the whole pattern}}{\text{number of units belonging to the whole pattern}}. \qquad (38)$$

By definition, we get

$$\rho'(t, \xi) = \int_1^{\xi^{D'}} n(t, s)\rho(s, \xi)\, \mathrm{d}s, \qquad (39)$$

where $n(t, s)$ is the number of t-unit subsegments in an s-unit segment. Self-similarity of segments leads to

$$n(t, s)/s \begin{cases} \propto t^{-(1+D'/D'')} & (l_s \gtrsim r_t), \\ =0 & (l_s \lesssim r_t), \end{cases} \qquad (40)$$

where l_s is the length of an s-unit segment given by (6) and r_t is that of a t-unit subsegment related to t as $r_t^{D''} \propto t$. Substituting (8) and (40) into (39), we have a consistent result

$$\rho'(t, \xi) \propto t^{-(1+D/D'')} \qquad (1 \ll t \lesssim \xi^{D''}). \qquad (41)$$

A geometrical (fractal) interpretation of thermal critical phenomena was presented by Suzuki (1983). He showed that $d - \beta/\nu$ can be regarded as a fractal dimensionality of the resultant dominant cluster which corresponds to typical segment size in our discussions. The present work supports this interpretation and makes geometrical meanings clearer. In addition, the distribution of clusters (droplets) near critical points turns out to be described by (4) and (8). Suzuki (1974) also proposed weak universality, that is, universality for the critical exponents $\hat{\alpha}$, $\hat{\beta}$ and $\hat{\gamma}$ defined through the correlation length ξ. It becomes evident that weak universality means universality for the fractal dimensionality of clusters (droplets) dominating critical behaviour. We consider that from a geometrical point of view, critical (non-analytic) behaviour of physical quantities generally comes from the fractal nature of systems.

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